

RELATIVE COHEN–MACAULAYNESS AND RELATIVE UNMIXEDNESS OF BIGRADED MODULES

MARYAM JAHANGIRI, AHAD RAHIMI

ABSTRACT. In this paper we study the finitely generated bigraded modules over a standard bigraded polynomial ring which are relative Cohen–Macaulay or relative unmixed with respect to one of the irrelevant bigraded ideals. A generalization of Reisner’s criterion for Cohen–Macaulay simplicial complexes is considered.

INTRODUCTION

Let $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be the standard bigraded polynomial ring over a field K and bigraded irrelevant ideals $P = (x_1, \dots, x_m)$ and $Q = (y_1, \dots, y_n)$. Let M be a finitely generated bigraded S -module. In [8] we call M to be relative Cohen–Macaulay with respect to Q if we have only one nonvanishing local cohomology with respect to Q . In other words, $\text{grade}(Q, M) = \text{cd}(Q, M)$ where $\text{cd}(Q, M)$ denote the cohomological dimension of M with respect to Q . Our aim in this paper is to investigate more about relative Cohen–Macaulay modules and its related topics like relative unmixedness. We organize this paper as follows: In Section 1, we first ask the following question:

Let M be relative Cohen–Macaulay with respect to P and Q . Is M itself Cohen–Macaulay? We have a counterexample which shows that the question is not true for dimension 2. Even though, for two given ideals I and J of a local ring R and a finitely generated R -module of M which is relative Cohen–Macaulay with respect to I and J , the question does not hold. We give some especial cases in which the question holds.

We call M to be relative unmixed with respect to Q if $\text{cd}(Q, M) = \text{cd}(Q, S/\mathfrak{p})$ for all $\mathfrak{p} \in \text{Ass } M$. We show that relative Cohen–Macaulay modules with respect to Q are relative unmixed with respect to Q . The converse does not hold in general. In the case in which every quotient of M is relative unmixed with respect to Q then it holds. Next we change the above question in the following sense:

Let M is relative Cohen–Macaulay with respect to P and relative unmixed with respect to Q . Is M itself unmixed? The local version of this question is not the case for dimension 2. We prove that the question has positive answer in the following bigraded cases: M be a bigraded S -module for which i) $\text{cd}(P, M) \leq 1$ and $\text{cd}(Q, M) \geq 0$, ii) $M = M_1 \otimes_K M_2$ where M_1 is a graded $K[x]$ -module and M_2 is a graded $K[y]$ -module and where $S/(\mathfrak{p}_1 + \mathfrak{p}_2)S$ is an integral domain for all $\mathfrak{p}_1 \in \text{Ass } M_1$

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and $\mathfrak{p}_2 \in \text{Ass } M_2$, iii) every cyclic submodule of M is pure, iv) $M = S/I$ where I is a monomial ideal. We believe that the question has negative answer for dimension 4. Until now we are not succeed to find such a counterexample. We have this question at the end of this section.

In Section 2, we describe explicitly the krull-dimension of the graded components of local cohomology of relative Cohen–Macaulay modules. We show that if M is relative Cohen–Macaulay with respect to Q with $\text{cd}(Q, M) = q$, then $\dim_S H_Q^q(M) = p$ where $p = \text{cd}(P, M)$. More general is true for its graded components, namely if $f_Q(M) = \text{cd}(Q, M) = q$ and $p + q = \dim M$, then $\dim_{K[x]} H_Q^q(M)_j = p$ for $j \ll 0$ where $f_Q(M)$ is the finiteness dimension of M relative to Q . As a consequence, if M is relative Cohen–Macaulay with respect to Q , then $H_Q^q(M)$ is an Artinian S -module if and only if $q = \dim(M)$. In other words, $H_Q^q(M)$ is not Artinian unless the ordinary known case.

In the following section we consider the hypersurface ring $R = S/fS$ where f is a bihomogeneous element of S . We show that the local cohomologies $H_Q^i(R)$ for $i = n, n - 1$ where $n \geq 2$ are never finitely generated. Moreover, $H_Q^n(R)$ is an Artinian S -module for $m \leq 1$, and $H_Q^{n-1}(R)$ is an Artinian S -module if and only if $m = 0$.

In the final section, we let Δ be a simplicial complex on $[n + m]$ and $K[\Delta] = S/I_\Delta$ its Stanley–Reisner ring. We say that Δ is relative Cohen–Macaulay with respect to Q over K if $K[\Delta]$ is relative Cohen–Macaulay with respect to Q . We show that $\text{cd}(Q, K[\Delta]) = \dim \Delta_W + 1$ where Δ_W is the subcomplex of Δ whose faces are subsets of W . This generalizes the known fact that for every simplicial complex Δ one has $\dim K[\Delta] = \dim \Delta + 1$. Using this fact and the generalization Hochster’s formula [8] we prove the following: Δ is relative Cohen–Macaulay with respect to Q with $\text{cd}(Q, K[\Delta]) = q$ if and only if $\tilde{H}_i((\text{link } F \cup G)_W; K) = 0$ for all $F \in \Delta_W$, $G \subset V$ and all $i < \dim \text{link}_{\Delta_W} F$. This in particular implies the Reisner’s criterion for Cohen–Macaulay simplicial complexes. A general version of this statement for monomial case is obtained.

1. COHEN–MACAULAYNESS AND UNMIXEDNESS WITH RESPECT TO P , Q AND $P + Q$

In [8] we call M to be relative Cohen–Macaulay with respect to Q if $H_Q^i(M) = 0$ for all $i \neq q$ with $q \geq 0$. In other words, $\text{grade}(Q, M) = \text{cd}(Q, M)$ where $\text{cd}(Q, M)$ denote the cohomological dimension of M with respect to Q . We recall the following facts from [8] which will be used in the sequel.

$$(1) \quad \text{cd}(P, M) = \dim M/QM \quad \text{and} \quad \text{cd}(Q, M) = \dim M/PM.$$

It is natural to ask the following question:

Question 1.1. *Let (R, \mathfrak{m}) be a Noetherian local ring, I and J two ideals of R such that $I + J = \mathfrak{m}$ and M a finitely generated R -module. If M is relative Cohen–Macaulay with respect to I , i.e., $\text{grade}(I, M) = \text{cd}(I, M)$ and relative Cohen–Macaulay with respect to J , i.e., $\text{grade}(J, M) = \text{cd}(J, M)$. Is M itself Cohen–Macaulay?*

In the following, we give several examples which shows that the question is not the case in general for graded, local and bigraded cases.

Example 1.2. Consider the standard graded polynomial ring $S = K[x_1, \dots, x_{2n}]$ with $n \geq 1$ and $\deg x_i = 1$ for all i . Set $P = (x_1, \dots, x_n)$, $Q = (x_{n+1}, \dots, x_{2n})$ and $\mathfrak{m} = (x_1, \dots, x_{2n})$ the unique graded maximal ideal of S . Set $R = S \oplus S/\mathfrak{p}$ where $\mathfrak{p} = (x_1 + x_{n+1}, x_2 + x_{n+2}, \dots, x_n + x_{2n})$. One has that S/\mathfrak{p} is Cohen–Macaulay S -module of dimension n , $\text{depth } R = n$ and $\dim R = 2n$. On the other hand, $\text{grade}(P, R) = \text{cd}(P, R) = \text{grade}(Q, R) = \text{cd}(Q, R) = n$. Thus R is relative Cohen–Macaulay with respect to P and Q , but R itself is not Cohen–Macaulay. Localizing R at the maximal ideal \mathfrak{m} and note that for any graded ideal I of S we have $\text{grade}(I, R) = \text{grade}(I_{\mathfrak{m}}, R_{\mathfrak{m}})$, $\text{cd}(I, R) = \text{cd}(I_{\mathfrak{m}}, R_{\mathfrak{m}})$, $\text{depth}_S R = \text{depth}_{S_{\mathfrak{m}}} R_{\mathfrak{m}}$ and $\dim_S R = \dim_{S_{\mathfrak{m}}} R_{\mathfrak{m}}$. Now one easily deduces that the question is not the case in the local case too.

Example 1.3. Let $n \geq 2$, and let $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ be the standard bigraded polynomial ring with $\deg x_i = (1, 0)$ and $\deg y_i = (0, 1)$ for $i = 1, \dots, n$. Set $I = \bigcap_{i=1}^n \mathfrak{p}_i$ where $\mathfrak{p}_i = (x_i, y_i)$ for $i = 1, \dots, n$ and $R = S/I$. Let \mathfrak{m} be the unique graded maximal ideal of S . From the exact sequence $0 \rightarrow S/I \rightarrow \bigoplus_{i=1}^n S/\mathfrak{p}_i \rightarrow S/\mathfrak{m} \rightarrow 0$ we have the exact sequence

$$\rightarrow H_Q^j(S/I) \rightarrow \bigoplus_{i=1}^n H_Q^j(S/\mathfrak{p}_i) \rightarrow H_Q^j(S/\mathfrak{m}) \rightarrow H_Q^{j+1}(S/I) \rightarrow .$$

Note that $H_Q^0(S/\mathfrak{m}) = S/\mathfrak{m}$ and $H_Q^j(S/\mathfrak{p}_i) = 0$ for $j \neq n-1$ and all i . It follows that $\text{grade}(Q, R) = 1$ and $\text{cd}(Q, R) = n-1$. By a similar argument, applying the functor $H_P^i(-)$ to the above short exact sequence one obtains $\text{grade}(P, R) = 1$ and $\text{cd}(P, R) = n-1$. Therefore R is relative Cohen–Macaulay with respect to P or Q if and only if $n = 2$. On the other hand, one has $\text{depth } R = n-1$ and $\dim R = 2(n-1)$. Thus if $n = 2$, then R is relative Cohen–Macaulay with respect to P and Q , but not Cohen–Macaulay.

In the following we give two special cases in which the question holds. We recall the following theorem from [8]

Theorem 1.4. *Let M be a finitely generated bigraded S -module which is relative Cohen–Macaulay with respect to Q and $|K| = \infty$. Then we have $\text{cd}(Q, M) + \text{cd}(P, M) = \dim M$.*

Proposition 1.5. *Let M be a finitely generated bigraded S -module with $\text{cd}(P, M) = p$ and $\text{cd}(Q, M) = q$ and let $|K| = \infty$. The following statements hold:*

- (a) *if M is relative Cohen–Macaulay with respect to P and Q with $p = 0$ or $p = \dim M$ and $q \geq 0$. Then M is Cohen–Macaulay.*
- (b) *if $M = M_1 \otimes_K M_2$ where M_1 is finitely generated graded $K[x]$ -module and M_2 is finitely generated graded $K[y]$ -module. If M is relative Cohen–Macaulay with respect to P and Q , then M is Cohen–Macaulay.*

Proof. In order to proof (a) we consider the spectral sequence $H_Q^i(H_P^j(M)) \xRightarrow{i} H_{\mathfrak{m}}^{i+j}(M)$ where $\mathfrak{m} = P + Q$. As $H_P^j(M) = 0$ for all $j \neq 0$, then the above spectral sequence degenerates and one obtains for all i the following isomorphism of bigraded S -modules, $H_Q^i(H_P^0(M)) \cong H_{\mathfrak{m}}^i(M)$. Using the fact that $\text{cd}(P, M) = 0$ if and only if $H_Q^0(M) = M$, we therefore have $H_Q^i(M) \cong H_{\mathfrak{m}}^i(M)$. Since $H_Q^i(M) = 0$ for all $i \neq q$, it follows that $H_{\mathfrak{m}}^i(M) = 0$ for all $i \neq q$ and so M is Cohen–Macaulay. Now let $p = \dim M$. By Theorem 1.4, we have $q = 0$ and then by a similar proof as above M is Cohen–Macaulay.

In order to proof (b) we note that $H_P^i(M) \cong H_P^i(M_1) \otimes_K M_2$ for all i , see the proof [8, Proposition 1.5]. Since M is relative Cohen–Macaulay with respect to P , it follows that M_1 is Cohen–Macaulay of dimension p . In fact, since $H_P^p(M) \neq 0$, it follows that $H_P^p(M_1) \neq 0$ and so $p \leq \dim_{K[x]} M_1$. If $p < \dim_{K[x]} M_1$, then $0 = H_P^{\dim M_1}(M) \cong H_P^{\dim M_1}(M_1) \otimes_K M_2$. As M_2 is finitely generated faithful K -module, Grusen’s theorem implies that $H_P^{\dim M_1}(M_1) = 0$, a contradiction. Therefore $\dim_{K[x]} M_1 = p$. By a similar argument as above we have $\text{depth}_{K[x]} M_1 = p$. Similarly, from the isomorphism $H_Q^i(M) \cong M_1 \otimes_K H_Q^i(M_2)$ for all i we have $\text{depth}_{K[y]} M_2 = \dim_{K[y]} M_2 = q$. By [11, Corollary 2.3] we have $\text{depth } M = \dim M = p + q$, as desired. \square

Remark 1.6. By Proposition 1.5(a), we deduce that Question 1.1 has positive answer while Example 1.3 shows that the question has negative answer when $\dim M = 2$.

We recall the following known facts which will be used in the rest of paper:

Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of S -modules with M finitely generated, then

$$(2) \quad \text{cd}(Q, M) = \max\{\text{cd}(Q, M'), \text{cd}(Q, M'')\},$$

Let $\text{Min } M$ denote the minimal elements of $\text{Supp } M$, then

$$(3) \quad \text{cd}(Q, M) = \max\{\text{cd}(Q, S/\mathfrak{p}) : \mathfrak{p} \in \text{Ass}(M)\}.$$

Note also that

$$\begin{aligned} \text{cd}(Q; M) &= \max\{\text{cd}(Q, S/\mathfrak{p}) : \mathfrak{p} \in \text{Supp}(M)\} \\ &= \max\{\text{cd}(Q, S/\mathfrak{p}) : \mathfrak{p} \in \text{Min}(M)\}. \end{aligned}$$

Proposition 1.7. *Let M be a finitely generated bigraded S -module with $|K| = \infty$, then we have*

$$\text{grade}(Q, M) \leq \text{cd}(Q, S/\mathfrak{p}) \quad \text{for all } \mathfrak{p} \in \text{Ass}(M).$$

Proof. Here we follow the proof of [2, Proposition 1.2.13]. Let $\mathfrak{p} \in \text{Ass } M$. We proceed by induction on $\text{grade}(Q, M)$. The claim is obvious if $\text{grade}(Q, M) = 0$. Now let $\text{grade}(Q, M) = k > 0$ and suppose inductively that the result has been proved for all finitely generated bigraded S -module N such that $\text{grade}(Q, N) < k$. We want to prove it for M . Since $\text{grade}(Q, M) > 0$, by [8, Lemma 3.4] there exists a bihomogeneous M -regular element $y \in Q$ which does not belong to any associated prime ideal of M and not to any minimal prime ideal of $\text{Supp}(M/PM)$ such that

$\text{cd}(Q, M/yM) = \text{cd}(Q, M) - 1$ and of course $\text{grade}(Q, M/yM) = \text{grade}(Q, M) - 1$. As in the proof of [2, Proposition 1.2.13] we see that \mathfrak{p} consists of zero divisors of M/yM . Thus $\mathfrak{p} \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \text{Ass}(M/yM)$. Since y is M -regular, it follows that $y \notin \mathfrak{p}$ while $y \in \mathfrak{q}$ and so $\mathfrak{p} \neq \mathfrak{q}$. Note also that, as y is M -regular and $\mathfrak{p} \in \text{Ass}(M)$, we have that y is S/\mathfrak{p} -regular and so $\text{grade}(Q, S/\mathfrak{p}) > 0$. Hence $\text{cd}(Q, S/\mathfrak{p}) = \dim S/(P + \mathfrak{p}) > 0$ by (1). We claim that the element y may be chosen to avoid all the minimal prime ideal of $\text{Supp}(S/(P + \mathfrak{p}))$, too. Let $\{\mathfrak{q}_1, \dots, \mathfrak{q}_r\}$ be the minimal prime ideals of $\text{Supp}(S/(P + \mathfrak{p}))$. By [8, Lemma 3.3] it suffices to show that $Q \not\subseteq \mathfrak{q}_i$ for $i = 1, \dots, r$. Suppose $Q \subseteq \mathfrak{q}_i$ for some i where $i = 1, \dots, r$. Since $P + \mathfrak{p} \subseteq \mathfrak{q}_i$, it follows that $\mathfrak{q}_i = P + Q = \mathfrak{m}$, and hence $\dim S/(P + \mathfrak{p}) = \text{cd}(Q, S/\mathfrak{p}) = 0$, a contradiction. Using inductive hypothesis and the above observation we have

$$\begin{aligned} \text{grade}(Q, M) - 1 &= \text{grade}(Q, M/yM) \\ &\leq \text{cd}(Q, S/\mathfrak{q}) \\ &= \dim S/(P + \mathfrak{q}) \\ &< \dim S/(P + \mathfrak{p}) = \text{cd}(Q, S/\mathfrak{p}), \end{aligned}$$

as desired. \square

This in particular generalizes the following known results

Corollary 1.8. *Let M be a finitely generated graded $K[y]$ -module, then we have*

$$\text{depth } M \leq \dim S/\mathfrak{p} \quad \text{for all } \mathfrak{p} \in \text{Ass}(M).$$

In particular, $\text{depth } M \leq \dim M$.

Corollary 1.9. *Let M be a finitely generated bigraded S -module, then we have*

$$\text{grade}(Q, M) \leq \text{cd}(Q, M).$$

Proof. The assertion follows from Proposition 1.7 and (3). \square

Definition 1.10. Let M be a finitely generated bigraded S -module. We call M to be relative unmixed with respect to Q if $\text{cd}(Q, M) = \text{cd}(Q, S/\mathfrak{p})$ for all $\mathfrak{p} \in \text{Ass}(M)$.

In the following we observe that relative Cohen–Macaulay modules with respect to Q are relative unmixed with respect to Q . In particular, all associated prime ideals of M are minimal in $\text{Supp } M/PM$.

Corollary 1.11. *Let M be a finitely generated bigraded S -module which is relative Cohen–Macaulay with respect to Q , then M is relative unmixed with respect to Q .*

Proof. By Proposition 1.7, we have $\text{grade}(Q, M) \leq \text{cd}(Q, S/\mathfrak{p})$ for all $\mathfrak{p} \in \text{Ass}(M)$. On the other hand, since $\mathfrak{p} \in \text{Ass}(M)$, we have the monomorphism $S/\mathfrak{p} \rightarrow M$ which yields $\text{cd}(Q, S/\mathfrak{p}) \leq \text{cd}(Q, M)$ by (2). Thus the conclusion follows. \square

Remark 1.12. Relative unmixed modules with respect to Q need not to be relative Cohen–Macaulay with respect to Q . We consider the hypersurface ring $R = S/fS$ where $f \in S$ is a bihomogeneous polynomial of degree (a, b) with $a, b > 0$ and f is not monomial as well. Note that $\text{Ass}(R) = \{(f)\}$. One has $\text{grade}(Q, R) = n - 1$ and $\text{cd}(Q, R) = n$. Thus R is relative unmixed with respect to Q but not relative Cohen–Macaulay with respect to Q .

The converse of Corollary 1.11 holds under the following additional assumption.

Proposition 1.13. *Let M be a finitely generated bigraded S -module for which every quotient of M is relative unmixed with respect to Q . Then M is relative Cohen–Macaulay with respect to Q .*

Proof. We proceed by induction on $q = \text{cd}(Q, M)$. The claim is obvious for $q = 0$. Assume $q > 0$ and the result has been proved for all finitely generated bigraded S -module of cohomological dimension less than q . We may assume that $\text{grade}(Q, M) > 0$. Otherwise, $Q \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \text{Ass}(M)$. Since M is relative unmixed with respect to Q , we have $0 < q = \text{cd}(Q, S/\mathfrak{p}) = \dim S/(P + \mathfrak{p}) \leq \dim S/(P + Q) = 0$, a contradiction. By [7, Lemma 3.4] there exists an M -regular bihomogeneous element $y \in Q$ such that $\text{cd}(Q, M/yM) = \text{cd}(Q, M) - 1$ as well as $\text{grade}(Q, M/yM) = \text{grade}(Q, M) - 1$. Our assumption implies that M/yM is relative unmixed with respect to Q and hence our induction hypothesis says that M/yM is relative Cohen–Macaulay with respect to Q . Therefore, M is relative Cohen–Macaulay with respect to Q , as desired. \square

The following question arises from Question 1.1:

Question 1.14. *Let (R, \mathfrak{m}) be a Noetherian local ring, I and J two ideals of R such that $I + J = \mathfrak{m}$ and M a finitely generated R -module. If M is relative Cohen–Macaulay with respect to I and relative unmixed with respect to J . Is M itself unmixed?*

Remark 1.15. In Example 1.2, we note that \mathfrak{p} is the only associated prime S/\mathfrak{p} and so $\text{Ass}(R) = \{\mathfrak{p}, (0)\}$. We have $\dim S/\mathfrak{p} = n < \dim R = 2n$ while R is relative Cohen–Macaulay with respect to P and Q . Therefore the question does not hold for $\dim M = 2$.

In the following we give several cases in which the Question 1.14 holds.

Proposition 1.16. *Let $|K| = \infty$ and let M be relative Cohen–Macaulay with respect to P with $\text{cd}(P, M) = p \leq 1$ or $p = \dim M$ and relative unmixed with respect to Q with $\text{cd}(Q, M) = q \geq 0$. Then M is unmixed.*

Proof. Let $\mathfrak{p} \in \text{Ass } M$. We first assume that $p = 0$ and so $\text{cd}(P, M) = \text{cd}(P, S/\mathfrak{p}) = 0$. Hence Theorem 1.4 yields $\text{cd}(Q, M) = \dim M$ and $\text{cd}(Q, S/\mathfrak{p}) = \dim S/\mathfrak{p}$. Therefore relative unmixedness of M with respect to Q results that M is unmixed. Now let $p = 1$ and so $\text{cd}(P, M) = \text{cd}(P, S/\mathfrak{p}) = 1$. We claim that S/\mathfrak{p} is relative Cohen–Macaulay with respect to P . Assume $\text{grade}(P, S/\mathfrak{p}) = 0$. The exact sequence $0 \rightarrow S/\mathfrak{p} \rightarrow M$ yields the exact sequence $0 \rightarrow H_P^0(S/\mathfrak{p}) \rightarrow H_P^0(M)$ and hence $\text{grade}(P, M) = 0$, a contradiction. By Theorem 1.4 we have

$$\dim M = \text{cd}(P, M) + \text{cd}(Q, M) = \text{cd}(P, S/\mathfrak{p}) + \text{cd}(Q, S/\mathfrak{p}) = \dim S/\mathfrak{p}.$$

The last equality follows again from Theorem 1.4. Finally, we assume that $p = \dim M$. Theorem 1.4 yields $q = 0$ and hence by a similar argument as the first part, M is unmixed. \square

Corollary 1.17. *Let $\dim M \leq 3$ and $|K| = \infty$. If M is relative Cohen–Macaulay with respect to P and relative unmixed with respect to Q . Then M is unmixed.*

Proposition 1.18. *Let M_1 and M_2 be two non zero finitely generated graded module over $K[x]$ and $K[y]$, respectively and let $|K| = \infty$. Set $M = M_1 \otimes_K M_2$ and assume that $K[x]/\mathfrak{p}_1 \otimes_K K[y]/\mathfrak{p}_2$ is an integral domain for all $\mathfrak{p}_1 \in \text{Ass } M_1$ and $\mathfrak{p}_2 \in \text{Ass } M_2$. If M is relative Cohen–Macaulay with respect to P and relative unmixed with respect to Q , then M is unmixed.*

Proof. Let $\mathfrak{p} \in \text{Ass}(M)$. Note that

$$\text{Ass}_S(M) = \bigcup_{\mathfrak{p}_1 \in \text{Ass}_{K[x]}(M_1)} \bigcup_{\mathfrak{p}_2 \in \text{Ass}_{K[y]}(M_2)} \text{Ass}_S(K[x]/\mathfrak{p}_1 \otimes_K K[y]/\mathfrak{p}_2),$$

see [11, Corollary 3.7]. Thus there exist $\mathfrak{p}_1 \in \text{Ass}_{K[x]}(M_1)$ and $\mathfrak{p}_2 \in \text{Ass}_{K[y]}(M_2)$ such that $\mathfrak{p} \in \text{Ass}_S(K[x]/\mathfrak{p}_1 \otimes_K K[y]/\mathfrak{p}_2) = \text{Ass}(S/\mathfrak{p}_1 S + \mathfrak{p}_2 S)$. By our assumption $S/(\mathfrak{p}_1 S + \mathfrak{p}_2 S)$ is an integral domain and so $\text{Ass}(S/\mathfrak{p}_1 S + \mathfrak{p}_2 S) = \{\mathfrak{p}_1 S + \mathfrak{p}_2 S\}$. Hence $\mathfrak{p} = \mathfrak{p}_1 + \mathfrak{p}_2$. Since M is relative Cohen–Macaulay with respect to P , it follows that M is relative unmixed with respect to P and so we have

$$\text{cd}(P, M) = \text{cd}(P, S/\mathfrak{p}) = \dim S/(Q + \mathfrak{p}) = \dim S/(Q + \mathfrak{p}_1) = \dim K[x]/\mathfrak{p}_1.$$

On the other hand, since M is relative unmixed with respect to Q , we have

$$\text{cd}(Q, M) = \text{cd}(Q, S/\mathfrak{p}) = \dim S/(P + \mathfrak{p}) = \dim S/(P + \mathfrak{p}_2) = \dim K[y]/\mathfrak{p}_2.$$

Thus by Theorem 1.4 and [11, Corollary 2.3], we have

$$\dim M = \text{cd}(P, M) + \text{cd}(Q, M) = \dim K[x]/\mathfrak{p}_1 + \dim K[y]/\mathfrak{p}_2 = \dim S/\mathfrak{p},$$

as desired. \square

Proposition 1.19. *Let M be a finitely generated bigraded S -module such that every cyclic submodule of M is pure. Let $|K| = \infty$ and assume M is relative Cohen–Macaulay with respect to P with $\text{cd}(P, M) = p$ and relative unmixed with respect to Q with $\text{cd}(Q, M) = q$. Then, M is unmixed.*

Proof. Let $\mathfrak{p} \in \text{Ass}(M)$. We claim that S/\mathfrak{p} is relative Cohen–Macaulay with respect to P . Let f_1, \dots, f_p be a maximal M -sequence in P . Since S/\mathfrak{p} is a cyclic submodule of M , the exact sequence $0 \rightarrow S/\mathfrak{p} \rightarrow M$ yields the exact sequence $0 \rightarrow S/(\mathfrak{p} + (f_1, \dots, f_p)) \rightarrow M/(f_1, \dots, f_p)M$. Since $f_i \notin Z(M/(f_1, \dots, f_{i-1})M)$ for all $i = 1, \dots, p$, it follows that $f_i \notin Z(S/(\mathfrak{p} + (f_1, \dots, f_{i-1})))$ for all $i = 1, \dots, p$. Thus f_1, \dots, f_p is an S/\mathfrak{p} -sequence in P which may not be maximal. Hence $\text{grade}(P, S/\mathfrak{p}) \geq p$. On the other hand, relative Cohen–Macaulayness of M with respect to P results that M is relative unmixed with respect to P and we have $\text{cd}(P, M) = \text{cd}(P, S/\mathfrak{p}) = p$. Thus $\text{grade}(P, S/\mathfrak{p}) \geq p = \text{cd}(P, S/\mathfrak{p})$. We conclude that $\text{grade}(P, S/\mathfrak{p}) = \text{cd}(P, S/\mathfrak{p}) = p$ and so S/\mathfrak{p} is relative Cohen–Macaulay with respect to P . Using Theorem 1.4 we have

$$\dim M = \text{cd}(P, M) + \text{cd}(Q, M) = \text{cd}(P, S/\mathfrak{p}) + \text{cd}(Q, S/\mathfrak{p}) = \dim S/\mathfrak{p},$$

as desired. \square

Proposition 1.20. *Let $I \subseteq S$ be a monomial ideal and set $R = S/I$ with $|K| = \infty$. Assume that R is relative Cohen–Macaulay with respect to P with $\text{cd}(P, R) = p$ and relative unmixed with respect to Q with $\text{cd}(Q, R) = q$. Then R is unmixed.*

Proof. Let $\mathfrak{p} \in \text{Ass}(R)$. By Corollary 1.4, we have $\text{cd}(P, R) = \text{cd}(P, S/\mathfrak{p}) = p$ and by our assumption $\text{cd}(Q, R) = \text{cd}(Q, S/\mathfrak{p}) = q$. Note that, the associated prime ideals of a monomial ideal are monomial prime ideals, see [5, Corollary 1.3.9]. The equality $\dim S/(Q + \mathfrak{p}) = p$ guaranties the existence $x_{i_{p+1}}, \dots, x_{i_m} \in \mathfrak{p}$ for which $x_{i_1}, \dots, x_{i_p} \notin \mathfrak{p}$ where $x_{i_1}, \dots, x_{i_m} \in \{x_1, \dots, x_m\}$ for all i . On the other hand, the equality $\dim S/(P + \mathfrak{p}) = q$ guaranties the existence $y_{j_{q+1}}, \dots, y_{j_n} \in \mathfrak{p}$ for which $y_{i_1}, \dots, y_{i_q} \notin \mathfrak{p}$ where $y_{j_1}, \dots, y_{j_n} \in \{y_1, \dots, y_n\}$ for all i . Thus, we conclude that $\mathfrak{p} = (x_{i_{p+1}}, \dots, x_{i_m}, y_{j_{q+1}}, \dots, y_{j_n})$. Therefore $\dim S/\mathfrak{p} = p + q = \dim R$ which follows from Theorem 1.4. \square

Remark 1.21. Let M be a relative Cohen–Macaulay with respect to Q with $\text{cd}(Q, M) = q$ and relative unmixed with respect to P with $\text{cd}(P, M) = p$ for which M is unmixed. Then all the associated prime ideals of M have the same height, namely $n + m - (p + q)$.

In Corollary 1.17, we observed that the Question 1.14 holds for $\dim M \leq 3$. We end this section with the following question:

Question 1.22. Let M be a finitely generated bigraded S -module of dimension 4 which is relative Cohen–Macaulay with respect to P and Q . Is the module M unmixed?

2. THE KRULL-DIMENSION OF THE GRADED COMPONENTS OF LOCAL COHOMOLOGY OF RELATIVE COHEN–MACAULAY MODULES

In this section we describe explicitly the krull-dimension of the graded components of local cohomology of relative Cohen–Macaulay modules. As a first result we have the following

Proposition 2.1. *Let M be a finitely generated bigraded S -module with $\text{cd}(P, M) = p$, $\text{cd}(Q, M) = q$ and $|K| = \infty$. The following statements hold:*

- (a) *if M is relative Cohen–Macaulay with respect to Q , then $\dim_S H_Q^q(M) = p$,*
- (b) *if M is relative Cohen–Macaulay with respect to P , then $\dim_S H_P^p(M) = q$.*

Proof. In order to prove (a) we note that $\text{Supp } H_Q^q(M) \subseteq \text{Supp } M/QM$. Thus we have $\dim H_Q^q(M) \leq \dim M/QM = \text{cd}(P, M) = p$. Since M is relative Cohen–Macaulay with respect to Q , from the spectral sequence $H_P^i(H_Q^j(M)) \xrightarrow{i} H_m^{i+j}(M)$ we get the following isomorphisms of bigraded S -modules $H_P^i(H_Q^q(M)) \cong H_m^{i+q}(M)$ for all i . By Theorem 1.4 we have $p + q = \dim M$ which yields $H_P^p(H_Q^q(M)) \neq 0$ and $H_P^i(H_Q^q(M)) = 0$ for $i > p$. Thus, we conclude that $\text{cd}(P, H_Q^q(M)) = p$, which is always less than or equal $\dim H_Q^q(M)$. Therefore $\dim_S H_Q^q(M) = p$. Part (b) is proved in the same way. \square

Let M be a finitely generated bigraded S -module. Recall the finiteness dimension of M relative to Q by:

$$f_Q(M) = \inf\{i \in \mathbb{N} : H_Q^i(M) \text{ is not finitely generated}\}.$$

Proposition 2.2. *Let M be a finitely generated bigraded S -module with $\text{cd}(P, M) = p$, $\text{cd}(Q, M) = q$ and $p + q = \dim M$. Then the following statements hold:*

- (a) *if $f_Q(M) = \text{cd}(Q, M) = q$, then $\dim_{K[x]} H_Q^q(M)_j = p$ for $j \ll 0$.*
- (b) *if $f_P(M) = \text{cd}(P, M) = p$, then $\dim_{K[y]} H_P^p(M)_j = q$ for $j \ll 0$.*

Proof. For the proof (a), we consider the spectral sequence $H_P^i(H_Q^k(M))_{(*,j)} \xRightarrow{i} H_{\mathfrak{m}}^{i+k}(M)_{(*,j)}$. Observe that $H_P^i(H_Q^k(M))_{(*,j)} = H_{P_0}^i(H_Q^k(M)_{(*,j)})$ where P_0 is the graded maximal ideal of $K[x]$. This equality follows from the definition of local cohomology using the Čech complex. Note that $H_Q^k(M)_j = 0$ for all $k < \text{cd}(Q, M) = q$ and $j \ll 0$. Thus the spectral sequence degenerates and one obtains for all i and $j \ll 0$ the following isomorphisms of bigraded $K[x]$ -modules $H_{P_0}^i(H_Q^q(M)_{(*,j)}) \cong H_{\mathfrak{m}}^{i+q}(M)_{(*,j)}$. Since $H_{\mathfrak{m}}^{p+q}(M)$ is a non zero Artinian S -module which is not finitely generated, it follows that $H_{\mathfrak{m}}^{p+q}(M)_j \neq 0$ for $j \ll 0$. Thus $H_{P_0}^p(H_Q^q(M)_j) \neq 0$ for $j \ll 0$ while $H_{P_0}^i(H_Q^q(M)_j) = 0$ for $i > p$. Therefore $\dim_{K[x]} H_Q^q(M)_j = p$ for $j \ll 0$, as desired. Part (b) is proved in the same way. \square

Corollary 2.3. *Let M be a finitely generated bigraded S -module with $f_Q(M) = \text{cd}(Q, M) = q$, $p + q = \dim M$ and $|K| = \infty$. Then $H_Q^q(M)$ is an Artinian S -module if and only if $q = \dim(M)$.*

Proof. Assume that $H_Q^q(M)$ is an Artinian S -module. Then, one has that $H_Q^q(M)_j$ is an Artinian $K[x]$ -module for all j . Therefore $\dim_{K[x]} H_Q^q(M)_j = 0$ for all j . On the other hand, in view of Proposition 2.2(a) we have that $\dim_{K[x]} H_Q^q(M)_j = \text{cd}(P, M)$ for $j \ll 0$. Thus, we deduce that $\text{cd}(P, M) = 0$ and hence by Theorem 1.4 we have $q = \dim M$. The converse is a well known fact. \square

Corollary 2.4. *Let M be a finitely generated bigraded S -module with $|K| = \infty$. The following statements hold:*

- (a) *if M is relative Cohen–Macaulay with respect to Q , then $\dim_{K[x]} H_Q^q(M)_j = p$ for $j \ll 0$. Moreover, $H_Q^q(M)$ is an Artinian S -module if and only if $q = \dim(M)$.*
- (b) *if M is relative Cohen–Macaulay with respect to P , then $\dim_{K[y]} H_P^p(M)_j = q$ for $j \ll 0$. Moreover, $H_P^p(M)$ is an Artinian S -module if and only if $p = \dim(M)$.*

Proof. The assertion follows from Proposition 2.2, Theorem 1.4 and Corollary 2.3. \square

Recall the Q -finiteness dimension $f_{\mathfrak{m}}^Q(M)$ of M relative to \mathfrak{m} by

$$f_{\mathfrak{m}}^Q(M) = \inf\{i \in \mathbb{N}_0 : Q \not\subseteq \text{rad}(0 : H_{\mathfrak{m}}^i(M))\}.$$

In view of [9, Proposition 2.3] one has

$$f_{\mathfrak{m}}^Q(M) = \sup\{i \in \mathbb{N}_0 : H_{\mathfrak{m}}^k(M)_j = 0 \text{ for all } k < i \text{ and all } j \ll 0\}.$$

Proposition 2.5. *Let M be a finitely generated bigraded S -module with $\text{cd}(P, M) = p$, $\text{cd}(Q, M) = q$ and $p + q = \dim M$. The following statements hold:*

- (a) *if M is generalized Cohen–Macaulay with $f_Q(M) = \text{cd}(Q, M) = q$, then $\text{depth}_{K[x]} H_Q^q(M)_j = p$ for $j \ll 0$.*
- (b) *if M is generalized Cohen–Macaulay with $f_P(M) = \text{cd}(P, M) = p$, then $\text{depth}_{K[y]} H_P^p(M)_j = q$ for $j \ll 0$.*

Proof. For the proof (a), since M is generalized Cohen–Macaulay, it follows that $f_m^Q(M) = \dim(M) = p + q$. By [4, Theorem 2.3] we have $\text{grade}(P_0, H_Q^q(M)_j) = f_m^Q(M) - \text{cd}(Q, M)$ for $j \ll 0$. This yields the desired claim. Part (b) is proved in the same way. \square

Corollary 2.6. *Let M be a finitely generated bigraded generalized Cohen–Macaulay S -module with $f_Q(M) = \text{cd}(Q, M) = q$ and $p + q = \dim M$. Then $H_Q^q(M)_j$ is Cohen–Macaulay $K[x]$ -module of dimension p for $j \ll 0$ and $\text{proj dim}_{K[x]} H_Q^q(M)_j = n - p$ for $j \ll 0$.*

3. FINITENESS PROPERTIES OF LOCAL COHOMOLOGY OF AN HYPERSURFACE RING

This is a well-known fact that the top local cohomology modules are almost never finitely generated. Let M be relative Cohen–Macaulay with respect to Q with $\text{cd}(Q, M) = q$. Thus $H_Q^q(M)$ is not finitely generated for $q > 0$. In Corollary 2.4 we observed that $H_Q^q(M)$ is not artinian as well, unless the ordinary known case $q = \dim M$. We consider the hypersurface ring $R = S/fS$ where $f \in S$ is a bihomogeneous form of degree (a, b) . This ring has only two nonvanishing local cohomology with respect to P or Q which is close to relative Cohen–Macaulay modules. In the following, we first observe that $H_Q^{n-1}(R)$ is not finitely generated, too for $n \geq 2$ and obtain some results on Artinianness of local cohomology of R .

Proposition 3.1. *Let $R = S/fS$ be a hypersurface ring. Then $H_Q^{n-1}(R)$ is not finitely generated for $n \geq 2$.*

Proof. The exact sequence $0 \rightarrow S(-a, -b) \xrightarrow{f} S \rightarrow S/fS \rightarrow 0$, induces the following exact sequence of S -modules

$$0 \rightarrow H_Q^{n-1}(R) \rightarrow H_Q^n(S)(-a, -b) \xrightarrow{f} H_Q^n(S) \rightarrow H_Q^n(R) \rightarrow 0.$$

Moreover, $H_Q^i(R) = 0$ for all $i < n - 1$. Let F be the quotient field of $K[x]$. Then

$$F \otimes_{K[x]} S = F[y_1, \dots, y_n] =: T.$$

Let T_+ be the graded maximal ideal of T . By the graded flat base change theorem, we have

$$F \otimes_{K[x]} H_Q^i(R) \cong H_{T_+}^i(F \otimes_{K[x]} R) \quad \text{for all } i.$$

Since $F \otimes_{K[x]} R = T/fT$ and $\dim T/fT = n - 1$, it follows that

$$H_{T_+}^i(T/fT) = 0 \quad \text{for all } i \neq n - 1.$$

Note that $H_{T_+}^{n-1}(T/fT)$ is an Artinian T -module which is not finitely generated. Thus $H_{T_+}^{n-1}(T/fT)_j \neq 0$ for all $j \ll 0$ and $n \geq 2$, and hence $H_Q^{n-1}(R)_j \neq 0$ for all $j \ll 0$ and $n \geq 2$. Therefore $H_Q^{n-1}(R)$ is not finitely generated for $n \geq 2$, as desired. \square

For bihomogeneous element $f \in S$, we denote by $c(f)$ the ideal of $K[x]$ generated by all the coefficients of f and $P_0 = (x_1, \dots, x_m)$ the graded maximal ideal of $K[x]$. A dual version of the above observation can be discussed as Artinianness of local cohomology of hypersurface rings.

Proposition 3.2. *Let $R = S/fS$ be a hypersurface ring. Then,*

- (a) *if $m \leq 1$, then $H_Q^n(R)$ is an Artinian S -module,*
- (b) *let $m \geq 2$. If $H_Q^n(R)$ is an Artinian S -module, then $c(f)$ is an P_0 -primary ideal which does not form a system of parameters for $K[x]$.*

Proof. For the proof (a) if $m = 0$, then Q is the graded maximal ideal of $K[y]$ and we may write $f = \sum_{|\beta|=b} c_\beta y^\beta$ where $c_\beta \in K$. Hence R is Cohen-Macaulay of dimension $n - 1$ and so $H_Q^n(R) = 0$ is Artinian. Let $m = 1$, we need to show $\text{Supp } H_Q^n(R) \subseteq \{\mathfrak{m}\}$ and $\text{Hom}(S/\mathfrak{m}, H_Q^n(R))$ is a finitely generated S -module where $\mathfrak{m} = P + Q$ is the unique graded maximal ideal of S . As $m = 1$, we may write $f = x^a \sum_{|\beta|=b} c_\beta y^\beta$ where $c_\beta \in K$. Thus $c(f) = (x^a)$ is an (x) -primary ideal and hence by [10, Corollary 2.6] we have $\text{Supp } H_Q^n(R) = \{\mathfrak{m}\}$. Since $m = 1$, by [3, Theorem 1] $H_Q^n(R)$ is Q -cofinite and so $\text{Hom}(S/Q, H_Q^n(R))$ is finitely generated. Therefore $\text{Hom}(S/\mathfrak{m}, H_Q^n(R))$ is finitely generated.

For the proof (b), as $H_Q^n(R)$ is an Artinian S -module, we have $\text{Supp } H_Q^n(R) \subseteq \{\mathfrak{m}\}$. By [10, Lemma 2.5] we have $\text{Supp } H_Q^n(R) = \{\mathfrak{q} \in \text{Spec } S : c(f) + Q \subseteq \mathfrak{q}\}$. Thus the maximal ideal \mathfrak{m} is the only minimal prime ideal $c(f) + Q$. It follows that P_0 is the only minimal prime ideal $c(f)$. Therefore $c(f)$ is an P_0 -primary ideal. Since $\text{Hom}(S/\mathfrak{m}, H_Q^n(R))$ is finitely generated, by [10, Theorem 2.3], $c(f)$ does not form a system of parameters for $K[x]$. \square

In the following, we show that $H_Q^{n-1}(R)$ is Artinian if and only if Q is the graded maximal ideal and $\dim R = n - 1$.

Proposition 3.3. *Let $R = S/fS$ be a hypersurface ring and $n \geq 1$. Then $H_Q^{n-1}(R)$ is an Artinian S -module if and only if $m = 0$.*

Proof. The exact sequence $0 \rightarrow S(-a, -b) \xrightarrow{f} S \rightarrow S/fS \rightarrow 0$, induces the following exact sequence of S -modules

$$0 \rightarrow H_Q^{n-1}(R) \rightarrow H_Q^n(S)(-a, -b) \xrightarrow{f} H_Q^n(S) \rightarrow H_Q^n(R) \rightarrow 0.$$

Note that $H_Q^{n-1}(R) = 0 \begin{smallmatrix} \vdots \\ H_Q^n(S) \end{smallmatrix} f \supseteq 0 \begin{smallmatrix} \vdots \\ H_Q^n(S) \end{smallmatrix} Q$ and $H_Q^n(S)$ is an Q -torsion S -module.

By [1, Theorem 7.1.2] we have $H_Q^{n-1}(R)$ is an Artinian S -module if and only if $H_Q^n(S)$ is an Artinian S -module. Hence by Corollary 2.4 this is equivalent to saying that $m = 0$. \square

4. GENERALIZATION OF REISNER'S CRITERION FOR COHEN-MACAULAY SIMPLICIAL COMPLEXES

As before, let $S = K[x_1, \dots, x_m, y_1, \dots, y_n]$ be the standard bigraded polynomial ring in $n + m$ variables over a field K and Δ a simplicial complex on $[n + m]$. We assume that Δ has vertices $\{v_1, \dots, v_m, w_1, \dots, w_n\}$ where vertices $V = \{v_1, \dots, v_m\}$ and $W = \{w_1, \dots, w_n\}$ correspond to the variables of x_1, \dots, x_m and y_1, \dots, y_n , respectively. We denote by Δ_W the restriction of Δ on W which is the subcomplex

$$\Delta_W = \{F \in \Delta : F \subseteq W\}.$$

Let F be a facet simplicial complex of Δ on $[n + m]$. We denote by \mathfrak{p}_F the prime ideal generated by all x_i and y_j such that $v_i, w_j \notin F$.

Proposition 4.1. *Let Δ be a simplicial complex on $[n + m]$ and $K[\Delta] = S/I_\Delta$ the Stanley-Reisner ring of Δ . Then*

$$\text{cd}(Q, K[\Delta]) = \dim \Delta_W + 1.$$

Proof. Using primary decomposition of $I_\Delta = \bigcap_F \mathfrak{p}_F$ where the intersection is taken over all facets of Δ , together with (1) and (3) we have

$$\begin{aligned} \text{cd}(Q, K[\Delta]) &= \max\{\text{cd}(Q, S/\mathfrak{q}) : \mathfrak{q} \in \text{Min}(K[\Delta])\} \\ &= \max\{\text{cd}(Q, S/\mathfrak{p}_F) : F \text{ is a facet of } \Delta\} \\ &= \max\{\dim S/(P + \mathfrak{p}_F) : F \text{ is a facet of } \Delta\} \\ &= \max\{\dim K[y_1, \dots, y_n]/\mathfrak{p}_G : G \text{ is a facet of } \Delta_W\} \\ &= \max\{|G| : G \text{ is a facet of } \Delta_W\} \\ &= \dim \Delta_W + 1, \end{aligned}$$

as required. \square

Corollary 4.2. *Let Δ be a simplicial complex on $[n]$, then one has $\dim K[\Delta] = \dim \Delta + 1$.*

We say that Δ is relative Cohen-Macaulay with respect to Q over K if $K[\Delta]$ is relative Cohen-Macaulay with respect to Q . We say that a simplicial complex Δ is pure if all facets have the same cardinality.

Corollary 4.3. *Let Δ is relative Cohen-Macaulay with respect to Q , then Δ_W is a pure simplicial complex.*

Proof. The assertion is immediate from Corollary 1.11 and Proposition 4.1. \square

Corollary 4.4. *Let $\dim \Delta_W = 0$, then Δ is relative Cohen-Macaulay with respect to Q .*

Proof. By Proposition 4.1 we have $\text{cd}(Q, K[\Delta]) = 1$. Since $\dim \Delta_W = 0$, it follows that the facets of Δ_W are the forms $F_i = (w_i)$ for $i = 1, \dots, n$ and hence $\mathfrak{p}_{F_i} = (x_1, \dots, x_m, y_1, \dots, \hat{y}_i, \dots, y_n)$ where $y_i \notin \mathfrak{p}_{F_i}$. Thus $Q \not\subseteq \mathfrak{q}$ for all $\mathfrak{q} \in \text{Ass}(K[\Delta])$. Therefore $\text{grade}(Q, K[\Delta]) = 1$, as required. \square

Let Δ be a simplicial complex on $[n]$. For a face F of Δ , the link of F in Δ is the subcomplex

$$\text{link}_\Delta F = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\},$$

and the star of F in Δ is the subcomplex

$$\text{star}_\Delta F = \{G \in \Delta : F \cup G \in \Delta\}.$$

Note that if Δ be a pure simplicial complex, then for any $F \in \Delta$ we have $\dim \text{link}_\Delta F = \dim \Delta - |F|$. We denote by $\tilde{H}_i(\Delta; K)$ the i th reduced homology group of Δ with coefficient in K , see Chapter 5 in [2] for details. We say that a simplicial complex Δ is connected if there exists a sequence of facets $F = F_0, \dots, F_t = G$ such that $F_i \cap F_{i+1} \neq \emptyset$ for $i = 0, \dots, t-1$. One has, Δ is connected if and only if $\tilde{H}_0(\Delta; K) = 0$. We set $\mathbb{Z}_-^m = \{a \in \mathbb{Z}^m : a_i \leq 0 \text{ for } i = 1, \dots, m\}$ and $\mathbb{Z}_+^n = \{b \in \mathbb{Z}^n : b_i \geq 0 \text{ for } i = 1, \dots, n\}$. We recall the following theorem from [7, Theorem 1.3].

Theorem 4.5. *Let $I \subset S$ be a squarefree monomial ideal. Then the bigraded Hilbert series of the local cohomology modules of $K[\Delta] = S/I$ with respect to the $\mathbb{Z}^m \times \mathbb{Z}^n$ -bigrading is given by*

$$\begin{aligned} H_{H_Q^i(K[\Delta])}(\mathbf{s}, \mathbf{t}) &= \sum_{a \in \mathbb{Z}_+^m, b \in \mathbb{Z}_-^n} \dim_K H_P^i(K[\Delta])_{(a,b)} \mathbf{s}^a \mathbf{t}^b \\ &= \sum_{F \in \Delta_W} \sum_{G \subset V} \dim_K \tilde{H}_{i-|F|-1}((\text{link } F \cup G)_W; K) \prod_{v_i \in G} \frac{s_i}{1-s_i} \prod_{w_j \in F} \frac{t_j^{-1}}{1-t_j^{-1}} \end{aligned}$$

where $\mathbf{s} = (s_1, \dots, s_m)$, $\mathbf{t} = (t_1, \dots, t_n)$, $G = \text{Supp } a$, $F = \text{Supp } b$ and Δ is the simplicial complex corresponding to the Stanley-Reisner ring $K[\Delta]$.

Here we note that $(\text{link } F \cup G)_W = \text{link}_{\Delta_W} F$. As an immediate consequence we obtain

Corollary 4.6. *We have $H_Q^i(K[\Delta])_{(a,b)} = 0$ for all i and for all $b \in \mathbb{Z}^n$ for which $b_j > 0$ for some j , or for all $a \in \mathbb{Z}^m$ for which $a_i < 0$ for some i and*

$$H_Q^i(K[\Delta])_{(a,b)} \cong \tilde{H}_{i-|F|-1}((\text{link } F \cup G)_W; K) \quad \text{for all } a \in \mathbb{Z}_+^m \text{ and all } b \in \mathbb{Z}_-^n,$$

where $G = \text{Supp } a$ and $F = \text{Supp } b$.

As a main result of this section we have the following. Here we follow the proof [5, Theorem 8.1.6].

Theorem 4.7. *Let Δ be a simplicial complex over a field K . The following conditions are equivalent.*

- (a) Δ is relative Cohen–Macaulay with respect to Q with $\text{cd}(Q, K[\Delta]) = q$,
- (b) $\tilde{H}_i((\text{link } F \cup G)_W; K) = 0$ for all $F \in \Delta_W$, $G \subset V$ and all $i < \dim \text{link}_{\Delta_W} F$.

Proof. Note that $\dim \Delta_W = q - 1$ by Proposition 4.1. Let Δ be relative Cohen–Macaulay with respect to Q . This is equivalent to saying that $H_Q^i(K[\Delta]) = 0$ for all $i \neq q$. Hence by Corollary 4.6, this is equivalent to saying that

$$(4) \quad \tilde{H}_{i-|F|-1}((\text{link } F \cup G)_W; K) = 0 \quad \text{for all } F \in \Delta_W, G \subset V \text{ and all } i < q.$$

(a) \implies (b): Since Δ is relative Cohen–Macaulay with respect to Q , by Corollary 4.3 it follows that Δ_W is pure and hence $\dim \text{link}_{\Delta_W} F = \dim \Delta_W - |F| = q - |F| - 1$. Therefore (4) implies that $\tilde{H}_i((\text{link } F \cup G)_W; K) = 0$ for all $F \in \Delta_W$, $G \subset V$ and all $i < \dim \text{link}_{\Delta_W} F$.

(b) \implies (a): Let $F \in \Delta_W$, $G \subset V$ and $H \in \text{link}_{\Delta_W} F$. Set $\Gamma = \text{link}_{\Delta_W} F$. One has

$$\text{link}_{\Gamma} H = \text{link}_{\Delta_W}(H \cup F) = (\text{link}(H \cup F \cup G))_W.$$

Hence our assumption yields

$$\tilde{H}_i(\text{link}_{\Gamma} H; K) = \tilde{H}_i((\text{link}(H \cup F \cup G))_W; K) = 0 \text{ for all } i < \dim \text{link}_{\Gamma} H.$$

Thus, by induction on the $\dim \Delta_W$ we may assume that all proper links of Δ_W are Cohen–Macaulay over K . In particular, the link of each vertex of Δ_W is pure. Thus all facets containing a given vertex have the same dimension. Now, let $\dim \Delta_W = 0$, by Corollary 4.4, Δ is relative Cohen–Macaulay with respect to Q . Thus we may assume that $\dim \Delta_W \geq 1$. Since $\tilde{H}_0(\Delta_W; K) = \tilde{H}_0(\text{link}_{\Delta_W} \emptyset; K) = 0$, it follows that Δ_W is connected. Thus Δ_W is a pure simplicial complex and hence for any $F \in \Delta_W$, we have $\dim \text{link}_{\Delta_W} F = q - |F| - 1$. Thus our hypothesis implies (4) and so Δ is relative Cohen–Macaulay with respect to Q . \square

As an immediate consequence we obtain the Reisner’s criterion for Cohen–Macaulay simplicial complexes

Corollary 4.8. *Let Δ be a simplicial complex and K a field. Then, Δ is Cohen–Macaulay over K if and only if $\tilde{H}_i(\text{link } F; K) = 0$ for all $F \in \Delta$ and all $i < \dim \text{link } F$.*

Proof. In Theorem 4.7 we assume that $m = 0$, then $G = \emptyset$, $(\text{link } F \cup G)_W = \text{link } F$, $\Delta_W = \Delta$ and Q is the unique maximal ideal \mathfrak{m} and $\text{cd}(Q, K[\Delta]) = \dim K[\Delta]$. \square

In the proof of the theorem we showed

Corollary 4.9. *Let Δ be relative Cohen–Macaulay with respect to Q , then Δ_W is connected.*

Corollary 4.10. *Let Δ be a relative Cohen–Macaulay complex with respect to Q and F is a face of Δ_W . Then $\text{link}_{\Delta_W} F$ is Cohen–Macaulay.*

Proof. The assertion follows from the beginning of the proof Theorem 4.7 (b) \implies (a) and Corollary 4.8. \square

Let $I \subset S$ be a monomial ideal and $G(I)$ the unique minimal monomial system of generators of I . For a monomial $u \in S$ we may write $u = u_1 u_2$ where $u_1 = x_1^{c_1} \dots x_m^{c_m}$ and $u_2 = y_1^{d_1} \dots y_n^{d_n}$. We set $\nu_i(u_1) = c_i$ for $i = 1, \dots, m$ and $\nu_j(u_2) = d_j$ for $j = 1, \dots, n$. We also set $\sigma_i = \max\{\nu_i(u_1) : u \in G(I)\}$ for $i = 1, \dots, m$ and $\rho_j = \max\{\nu_j(u_2) : u \in G(I)\}$ for $j = 1, \dots, n$. For $b = (b_1, \dots, b_n) \in \mathbb{Z}^n$ we set $G_b = \{j : 1 \leq j \leq n, b_j < 0\}$ and let $a \in \mathbb{Z}_+^m$. We define the simplicial complex $\Delta_{(a,b)}(I)$ whose faces are the set $L - G_b$ with $G_b \subseteq L$ and such that L satisfies the following conditions: for all $u \in G(I)$ there exists $j \notin L$ such that $\nu_j(u_2) > b_j \geq 0$, or for at least one i , $\nu_i(u_1) > a_i \geq 0$. We recall the following theorem from [7, Theorem 2.4].

Theorem 4.11. *Let $I \subset S$ be a monomial ideal. Then the Hilbert series of the local cohomology modules of S/I with respect to the $\mathbb{Z}^m \times \mathbb{Z}^n$ -bigrading is given by*

$$H_{H_Q^i(S/I)}(\mathbf{s}, \mathbf{t}) = \sum \sum \dim_K \tilde{H}_{i-|F|-1}(\Delta_{(a,b)}(I); K) \mathbf{s}^a \mathbf{t}^b,$$

where the first sum runs over all $F \in \Delta_W$, $b \in \mathbb{Z}^n$ for which $G_b = F$ and $b_j \leq \rho_j - 1$ for $j = 1, \dots, n$, and the second sum runs over all $a \in \mathbb{Z}^m$ for which $N_a = G$ and $a_i \geq \sigma_i - 1$ for $i = 1, \dots, m$. Here $N_a = \text{Supp } a$ and Δ is the simplicial complex corresponding to the Stanley-Reisner ideal \sqrt{I} .

The precise expression of the Hilbert series is given in [7]. As a first consequence we have

Corollary 4.12. *we have $H_Q^i(S/I)_{(a,b)} = 0$ for all i and for all $b \in \mathbb{Z}^n$ for which $b_j > \rho_j - 1$ for some j , or for all $a \in \mathbb{Z}^m$ for which $a_i < \sigma_i - 1$ for some i and*

$$H_Q^i(S/I)_{(a,b)} \cong \tilde{H}_{i-|F|-1}(\Delta_{(a,b)}(I); K)$$

for all $b \in \mathbb{Z}^n$ with $b_j \leq \rho_j - 1$ for $j = 1, \dots, n$, and $G_b = F$ and for all $a \in \mathbb{Z}^m$ with $a_i \geq \sigma_i - 1$ for $i = 1, \dots, m$ and $N_a = G$.

For a bigraded S -module M , we recall the a -invariant of M by

$$a_Q^i(M) = \sup\{\mu : H_Q^i(M)_{(*, \mu)} \neq 0\},$$

and so $\text{reg}(M) = \max_i \{a_Q^i(M) + i : i \geq 0\}$.

Corollary 4.13. *Suppose $I \subseteq S$ be a monomial ideal such that S/I is relative Cohen-Macaulay with respect to Q with $\text{cd}(Q, S/I) = q$. then*

$$\text{reg}(S/I) \leq \sum_{j=1}^n \rho_j - n + q.$$

Proof. Note that for all $k, j \in \mathbb{Z}$ we have

$$H_Q^q(S/I)_{(k,j)} = \bigoplus_{\substack{a \in \mathbb{Z}^m, |a|=k \\ b \in \mathbb{Z}^n, |b|=j}} H_Q^q(S/I)_{(a,b)},$$

where $|a| = \sum_{i=1}^m a_i$ for $a = (a_1, \dots, a_m)$ and $|b| = \sum_{i=1}^n b_i$ for $b = (b_1, \dots, b_n)$. By Corollary 4.12 we have that $H_Q^q(S/I)_{(k,j)} = 0$ for $k < \sum_{i=1}^m \sigma_i - m$ or $j > \sum_{j=1}^n \rho_j - n$. Thus we have

$$H_Q^q(S/I)_j = \bigoplus_k H_Q^q(S/I)_{(k,j)} = 0 \quad \text{for } j > \sum_{j=1}^n \rho_j - n.$$

Hence $a_Q^q(S/I) \leq \sum_{j=1}^n \rho_j - n$ and so the conclusion follows. \square

As a generalization of [6, Corollary 2.3] we have

Corollary 4.14. *Let $I \subset S$ be a monomial ideal. Then for all i we have the following isomorphisms of K -vector spaces*

$$H_Q^i(S/I)_{(a,b)} \cong H_Q^i(S/\sqrt{I})_{(a,b)},$$

for all $a \in \mathbb{Z}_+^m$ and $b \in \mathbb{Z}_-^n$. In particular, $\text{cd}(Q, S/I) = \text{cd}(Q, S/\sqrt{I})$.

Proof. By a similar proof as [6, Corollary 2.3] one has $\Delta_{(a,b)}(I) = \Delta_{(a,b)}(\sqrt{I})$. Thus Corollary 4.12 yields the desired isomorphism. \square

Now we come to a general version of Theorem 4.7 as follows:

Corollary 4.15. *Let $I \subseteq S$ be a monomial ideal and Δ the simplicial complex corresponding to \sqrt{I} . The following conditions are equivalent.*

- (a) S/I is relative Cohen–Macaulay with respect to Q with $\text{cd}(Q, S/I) = q$,
- (b) $\tilde{H}_i((\text{link } F \cup G)_W; K) = 0$ for all $F \in \Delta_W$, $G \subset V$ and all $i < \dim \text{link}_{\Delta_W} F$.

Proof. Note that

$$\Delta_{(a,b)}(I) = \Delta_{(a,b)}(\sqrt{I}) = \text{link}_{\text{star } N_a \cup H_b} G_b = \text{link}_{\text{star } N_a} G_b = (\text{link } F \cup G)_W,$$

see the remark after [7, Theorem 2.4] and also the proof [12, Corollary 1]. Now the assertion follows by applying Corollary 4.12 and Corollary 4.14 to Theorem 4.7. \square

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MARYAM JAHANGIRI, SCHOOL OF MATHEMATICS AND COMPUTER SCIENCE, DAMGHAN UNIVERSITY OF BASIC SCIENCES, DAMGHAN, IRAN. SCHOOL OF MATHEMATICS, INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM), P.O. BOX: 19395-5746, TEHRAN, IRAN.

E-mail address: jahangiri@dubs.ac.ir

AHAD RAHIMI, DEPARTMENT OF MATHEMATICS, RAZI UNIVERSITY, KERMANSHAH, IRAN.
SCHOOL OF MATHEMATICS, INSTITUTE FOR RESEARCH IN FUNDAMENTAL SCIENCES (IPM), P.
O. Box: 19395-5746, TEHRAN, IRAN.
E-mail address: `ahad.rahimi@razi.ac.ir`